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A Proof of Cauchy's Integral Theorem

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We give a simple proof of Cauchy's integral theorem viewed, as usual, as the foundation of complex analysis.

THEOREM. *Let f be holomorphic in an open set D of the (finite) complex plane. Let $c(t)$, $c_1(t)$ be complex functions, continuous and of bounded variation in* $[0, 1]$, *mapping it into D, and satisfying $c(0) = c(1)$, $c_1(0) = c_1(1)$. Suppose that c can be deformed in D (as a loop) to c_1, namely, suppose there exists a complex function $C(t, s)$, continuous in the square $0 \leqslant t \leqslant 1$, $0 \leqslant s \leqslant 1$, and mapping it into D such that*

$$C(0, s) = C(1, s), \qquad \textit{for every } s \in [0, 1],$$

$$C(t, 0) = c(t), \qquad \textit{and} \qquad C(t, 1) = c_1(t), \qquad \textit{for every } t \in [0, 1].$$

Then

$$\int_c f(z)\, dz = \int_{c_1} f(z)\, dz.$$

In particular, if c_1 is constant in $[0, 1]$ *so that c can be deformed in D to a point, then $\int_c f(z)\, dz = 0$.*

Proof. (I) We begin by assuming that f' is continuous in D and that $C(t, s)$ can be chosen so that its second order partial derivatives exist and are continuous at every point of $[0, 1] \times [0, 1]$.

For every $s \in [0, 1]$, let

$$I(s) = \int_{C(t,s)} f(z)\, dz = \int_0^1 f(C(t, s))\, dC(t, s) = \int_0^1 f(C(t, s)) \frac{\partial C}{\partial t}\, dt.$$

Then $I(s)$ is continuous in $[0, 1]$. We want to prove that $I(0) = I(1)$; this will follow if we show that $I'(s) = 0$ for every $s \in (0, 1)$.

But for every such s,

$$\begin{aligned} I'(s) &= \int_0^1 \left[f'(C(t,s)) \frac{\partial C}{\partial s} \frac{\partial C}{\partial t} + f(C(t,s)) \frac{\partial}{\partial s} \frac{\partial C}{\partial t} \right] dt \\ &= \int_0^1 \left[\frac{\partial}{\partial t} f(C(t,s)) \right] \frac{\partial C}{\partial s}\, dt + \int_0^1 f(C(t,s)) \frac{\partial}{\partial s} \frac{\partial C}{\partial t}\, dt \\ &= \left[f(C(t,s)) \frac{\partial C}{\partial s} \right]_{t=0}^{1} - \int_0^1 f(C(t,s)) \frac{\partial}{\partial t} \frac{\partial C}{\partial s}\, dt \\ &\quad + \int_0^1 f(C(t,s)) \frac{\partial}{\partial s} \frac{\partial C}{\partial t}\, dt = 0. \end{aligned}$$

(II) We drop now the assumption on C made in (I), but continue to assume f' is continuous in D. For $n = 1, 2, \ldots$, consider the Bernstein polynomial

$$B_n(t, s) \equiv \sum_{j=0}^{n} \sum_{k=0}^{n} C\left(\frac{j}{n}, \frac{k}{n}\right) \binom{n}{j} \binom{n}{k} t^j (1-t)^{n-j} s^k (1-s)^{n-k}.$$

Then [1, p. 122; or 2, p. 327] $B_n(t, s)$ converges uniformly to $C(t, s)$ in $[0, 1] \times [0, 1]$. As we easily see, there is an integer $n_0 \geqslant 1$, and a compact subset D_0 of D containing $C(t, s)$ and $B_n(t, s)$ whenever $n \geqslant n_0$, $0 \leqslant t \leqslant 1$, $0 \leqslant s \leqslant 1$.

Observe that, for $n = 1, 2, \ldots$,

$$\begin{aligned} B_n(0, s) &\equiv \sum_{k=0}^{n} C\left(0, \frac{k}{n}\right) \binom{n}{k} s^k (1-s)^{n-k} \\ &\equiv \sum_{k=0}^{n} C\left(1, \frac{k}{n}\right) \binom{n}{k} s^k (1-s)^{n-k} \equiv B_n(1, s). \end{aligned}$$

By part (I), for all $n \geqslant n_0$,

$$\int_{B_n(t,0)} f(z)\, dz = \int_{B_n(t,1)} f(z)\, dz. \tag{1}$$

Set

$$I_n = \int_{B_n(t,0)} f(z)\, dz - \int_c f(z)\, dz, \qquad n = n_0, n_0 + 1, n_0 + 2, \ldots \tag{2}$$

Then for these n we have, using Riemann–Stieltjes integration by parts,

$$I_n = \int_0^1 f(B_n(t,0))\,dB_n(t,0) - \int_0^1 f(B_n(t,0))\,dC(t,0)$$

$$+ \int_0^1 f(B_n(t,0))\,dC(t,0) - \int_0^1 f(C(t,0))\,dC(t,0)$$

$$= \int_0^1 [-B_n(t,0) + C(t,0)]\,df(B_n(t,0))$$

$$+ \int_0^1 [f(B_n(t,0)) - f(C(t,0))]\,dC(t,0);$$

$$|I_n| \leqslant \max_{0\leqslant t\leqslant 1} |B_n(t,0) - C(t,0)| \cdot \int_0^1 \left| f'\left(B_n(t,0)\right) \frac{d}{dt} B_n(t,0) \right| dt$$

$$+ [\max_{0\leqslant t\leqslant 1} |f(B_n(t,0)) - f(C(t,0))|]\, VC(t,0),$$

where $VC(t, 0)$ is the total variation of $C(t, 0)$ in $[0, 1]$.

Now, for $n = 1, 2, \ldots$,

$$B_n(t,0) \equiv \sum_{j=0}^{n} C\left(\frac{j}{n}, 0\right)\binom{n}{j} t^j (1-t)^{n-j}$$

is the (one-dimensional) nth order Bernstein polynomial of $C(t, 0)$. Therefore, by a simple property of these polynomials [4, p. 23],

$$\int_0^1 \left| \frac{d}{dt} B_n(t,0) \right| dt = VB_n(t,0) \leqslant VC(t,0), \qquad n = 1, 2, \ldots.$$

Hence, for $n \geqslant n_0$,

$$|I_n| \leqslant VC(t,0)[\max_{z\in D_0} |f'(z)| \cdot \max_{0\leqslant t\leqslant 1} |B_n(t,0) - C(t,0)|$$

$$+ \max_{0\leqslant t\leqslant 1} |f(B_n(t,0) - f(C(t,0))|] \to 0,$$

as $n \to \infty$ because of the uniform continuity of f on D_0. So, by (2),

$$\int_{B_n(t,0)} f(z)\,dz \to \int_c f(z)\,dz.$$

In the same way,

$$\int_{B_n(t,1)} f(z)\,dz \to \int_{c_1} f(z)\,dz.$$

Hence, by (1),

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz.$$

(III) Finally, the theorem under the additional hypothesis of continuity of f' in D readily implies the theorem without this restriction, as has been observed in the literature. For the restricted theorem implies, in the usual way, Cauchy's integral formula in, say, the form: If $F'(z)$ is continuous in an open set $E \ni z_0$, if $k(t)$ (with $k(0) = k(1)$) is a function, continuous and of bounded variation in $[0, 1]$, mapping it into $E - \{z_0\}$, and if, for some $\delta > 0$, k can be deformed in $E - \{z_0\}$ (as a loop) to every k_r, $0 < r < \delta$, where

$$k_r(t) \equiv z_0 + re^{2\pi i t}, \qquad 0 \leqslant t \leqslant 1,$$

then

$$F(z_0) = (2\pi i)^{-1} \int_k F(z)(z - z_0)^{-1}\,dz.$$

The last representation implies, in the usual manner, that a complex function possessing a continuous derivative throughout an open set has everywhere there derivatives of all orders. One proves also, say, by the usual method of bisection, that if F has a (finite) derivative at every point on or within a triangle T, then $\int_T F(z)\,dz = 0$. Suppose now that a function F is holomorphic in a disk $| z - a | < r$. The last result implies, by the standard argument, that F has there a primitive, namely $\int_{L_z} F(u)\,du$, where $L_z(t) \equiv a + t(z - a)$, $0 \leqslant t \leqslant 1$. This primitive, having a continuous derivative throughout the disk, has everywhere there derivatives of all orders; therefore, so does F. Hence, being holomorphic in an open set implies having everywhere there derivatives of all orders and, in particular, having there a continuous first derivative. Consequently, Cauchy's integral theorem holds in its unrestricted form, and similarly in Cauchy's integral formula the requirement that F' be continuous need not be explicitly made. We proved on our way some fundamental results other than Cauchy's integral theorem which should not be considered an extra effort if one's purpose is to construct complex analysis, as is customarily done, on the basis of Cauchy's integral theorem.

Remark. It is of interest to note, though not needed for our purpose, that the reasoning of (I) and (II) can be used to obtain a stronger result, namely, that the desired conclusion holds under the assumption that f' is bounded on every compact subset of D. Indeed, under this condition the formulas we derived in (I) for $I'(s)$ still hold; the integrals, however, being Lebesgue [3, Theorems 250, 242, 261]; in particular, again $I(0) = I(1)$. Similarly, the arguments of part (II) continue to hold, the only difference being that the integral involving $f'(B_n(t, 0))$ has to be considered as Lebesgue [5, p. 123, (iv)].

Note added in proof. The authors have just learned that the proof in (I) can be found in the literature, e.g. in Hurwitz-Courant, *Funktionentheorie*, Springer-Verlag, Berlin, 4th edition, 1964, p. 288.

References

1. P. J. Davis, "Interpolation and Approximation," Blaisdell Publishing Co., New York, 1963.
2. T. H. Hildebrandt and I. J. Schoenberg, On linear functional operations and the moment problem for a finite interval in one or several dimensions, *Ann. of Math.* (2), **34** (1933), 317–328.
3. H. Kestelman, "Modern Theories of Integration," 2nd revised edition, Dover Publications, New York, 1960.
4. G. G. Lorentz, "Bernstein Polynomials," University of Toronto Press, Toronto, 1953.
5. S. Saks, "Theory of the Integral," 2nd revised edition, Hafner Publishing Co., New York, 1937.